

Static Portfolio Optimization

I. Common Utility Functions

The most commonly used utility functions in financial economics belong to the hyperbolic absolute risk aversion (HARA) family.

$$u(x) = \tau \cdot \left(\mu + \frac{x}{\gamma} \right)^{1-\gamma}$$

Where τ, μ, γ are some numbers. Looks ugly doesn't it? Fortunately, we are only focusing on three special cases.

i. Quadratic Utility

$$u(x) = ax^2 + bx + c$$

Special property:

ii. Constant Absolute Risk Aversion (CARA)

$$u(x) = \frac{-e^{-Ax}}{A}$$

Where A is rate of absolute risk aversion.

iii. Constant Relative Risk Aversion (CRRA)

$$u(x) = \frac{x^{1-R}}{1-R}$$

Where R is rate of relative risk aversion.

What happens when $R = 1$?

II. Single Risky Asset Optimization

Let's consider the case where there are only two assets in this world—a risk-free, interest-bearing one, and a risky one. As you shall find out later on, this is in fact not a bad assumption. For simplicity, we shall call the riskless asset “bond” and the risky asset “stock”. Bond will have a sure return of r , while stock will have a risky return of $r + s$. Notice how we express the return of stock as the return of bond plus a number. In finance s is called *excess return*. E.g.

So if an investor invests x dollars in each, she would get in return,

Let us give our hypothetical investor a wealth of w , and suppose this investor invests θ of her wealth in stock. Then her final wealth is

Remember from your immediate microeconomics course, we maximize a utility function by taking its first derivative.

Simple? Unfortunately depending on how s looks like, the math is not necessarily easy. So as economists, we do what we always do—let's assume how s looks like!

Case ii. CARA

Assume S is *normally distributed* with mean μ and standard deviation σ .

What does a normal curve look like?

Properties: Let a be any number, $S + a$ is normal, $S * a$ is normal

Second concept: e^S is *log-normally distributed*.

What does log-normal mean?

What does a log-normal curve look like?

Properties: Let a be any number, $e^S * a$ is log-normal, $e^S + a$ is *shifted* log-normal

Actual Math

ONE RULE TO REMEMBER: If X is *log-normal*

$$\log(E[X]) = E[\log(X)] + \frac{1}{2} \text{Var}[\log(X)]$$

Example: $u(x) = \frac{-e^{-4x}}{4}$

III. Should we invest at all?

Notice from the solutions above. The investors always invest. This result actually holds for any utility function. Intuitively, why?

Why might we think this is not realistic?